Pre-class Warm-up

With reference to the picture on the right, what is the sum

$$\sum_{i=1}^{5} f(c_i)\cdot(x_i - x_{i-1})$$

called?

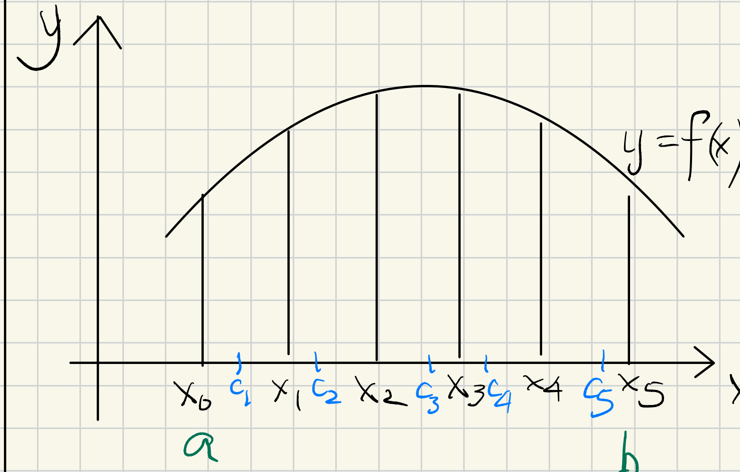
a. a Cauchy sum

b. a Newton sum

c. a Dedekind sum

d. a Riemann sum

e. Some other kind of sum



$$\int_a^b f(x)\,dx = \lim_{\text{strips get thinner}} \sum_i f(c_i)(x_i - x_{i-1})$$

Exam 1 is on Tuesday next week in the discussion session.

Sections 5.1 and 5.2: double integrals over rectangles

We learn

- Different notations for the double integral
- Interpretation as volume under the graph
- Interpretation as volume swept out by a slice (Cavalieri's principle)
- Proper definition using Riemann sums
- Some theoretical things: continuous implies integrable, bounded with restrict discontinuities implies integrable
- Fubini's theorem
- How to calculate integrals

Upshot:

The following three examples describe exactly the same integral.

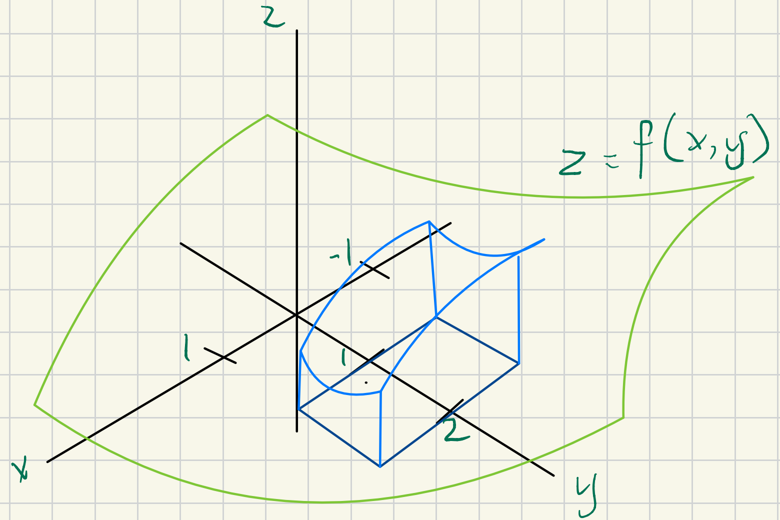
Examples: a. Find $\int_1^2 \left(\int_{-1}^{1} x^2y + y^3 \, dx \right) dy$

b. Find $\iint_R x^2y + y^3 \, dA$

where R is the rectangle [-1,1] x [1,2]

$= \{(x,y) \mid x \in [-1,1], y \in [1,2]\}$

c. Find the volume under the graph of f(x,y) = x^2y + y^3 above the rectangle [-1,1] x [1,2]

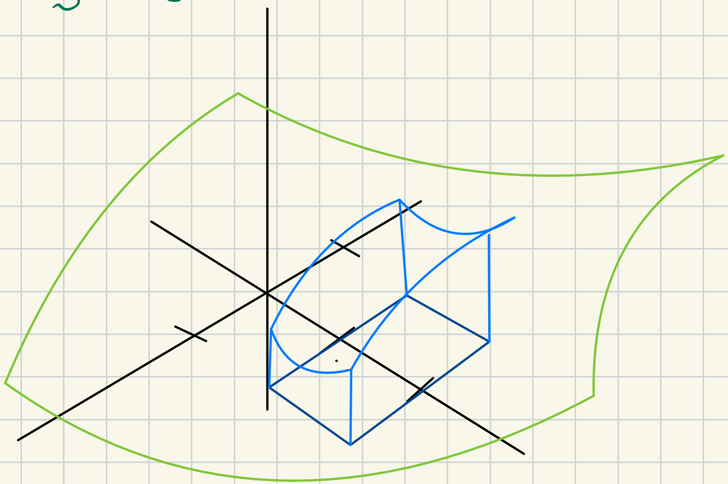


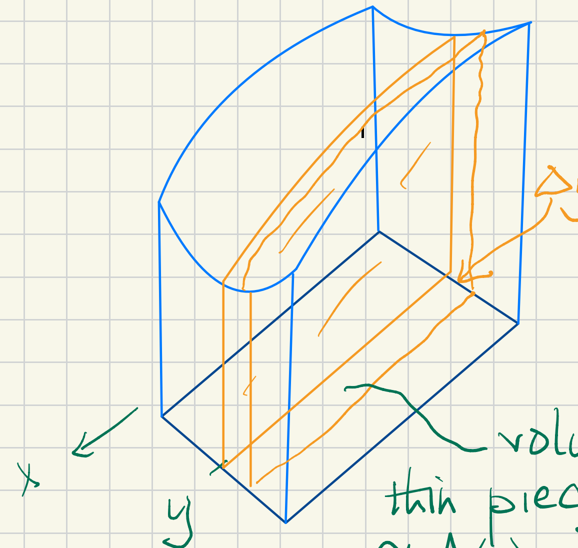
Examples: a. Find $\int_1^2 \left(\int_{-1}^1 x^2y + y^3 \, dx \right) dy$

Solution: We integrate first with respect to x to get $\int_1^2 \left[\frac{x^3 y}{3} + x y^3 \right]_{-1}^1 dy$

$$= \int_1^2 \left(\frac{2}{3} y + 2y^3 \right) dy = \left[\frac{y^2}{3} + \frac{1}{2} y^4 \right]_1^2$$

$$= \frac{3}{3} + \frac{15}{2} = 8\tfrac{1}{2}.$$



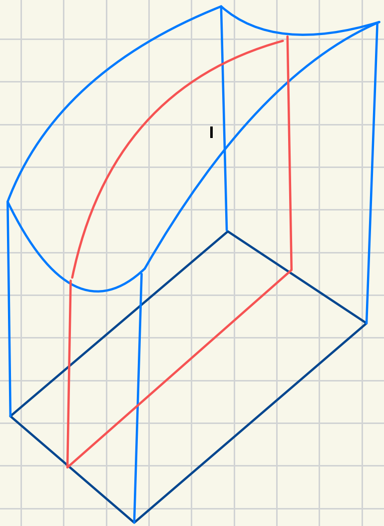


volume of thin piece is $\approx A(y) \cdot \Delta y$

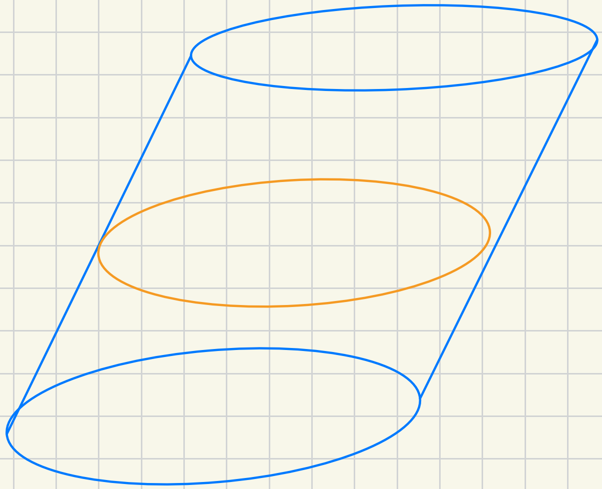
$A(y) = \int_{-1}^1 f(x,y)\, dx$ at y is the area under the graph of $f(x,y)$ with y fixed, as a function of x.

Adding the thin volumes $A(y)\Delta y$ gives approximately the volume under graph

Cavalieri's Principle



The volume of a solid is the integral of the cross-sectional area with respect to a coordinate pointing in another direction.

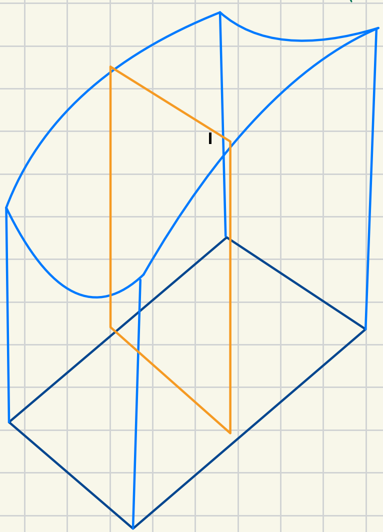


Examples: a. Find $\int_{-1}^{1}\left(\int_{1}^{2} x^2y + y^3 \, dy\right)dx$

$$= \int_{-1}^{1} \left[\frac{x^2y^2}{2} + \frac{y^4}{4}\right]_1^2 dx$$

$$= \int_{-1}^{1} \left(\frac{3x^2}{2} + \frac{15}{4}\right) dx = \left[\frac{x^3}{2} + \frac{15x}{4}\right]_{-1}^{1}$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{15}{4} + \frac{15}{4} = 8\tfrac{1}{2}.$$



We first find the area of the orange lamina.

Informal Fubini's theorem.

$$\int\int f \, dx\, dy = \int\int f \, dy\, dx.$$

What's wrong with this?
We don't yet know what we mean by the volume under the graph.
We don't have a proper definition of the integral.

Question:

What is $$\int_1^2 \int_{-1}^1 dx\,dy \quad ?$$

$= \text{area of } [-1,1]\times[1,2]$

$= 2$

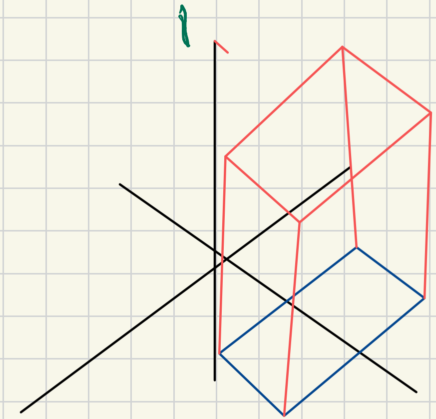
a. 0

b. 1

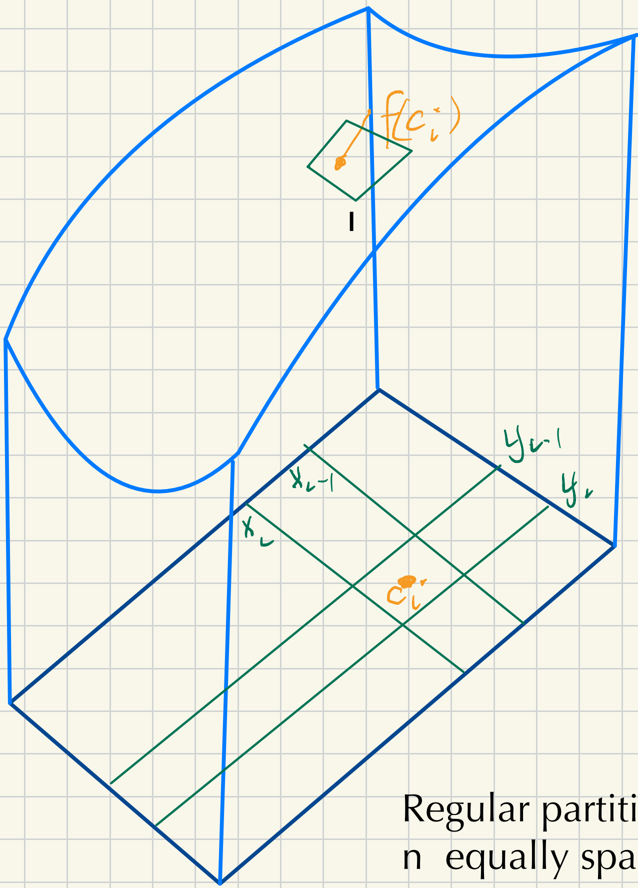
c. 2

d. 4

e. 1/2



Riemann sums



Regular partition:
n equally spaced points.

The Riemann sum is

$$\sum_{\substack{\text{everything} \\ i, j}} f(c_i)(x_i - x_{i-1})(y_j - y_{j-1})$$

We say the function f(x,y) is integrable if

as the $x_i - x_{i-1}$, $y_j - y_{j-1} \to 0$

The Riemann sums $\to$ some definite number

allowing for all possible c_i

This number is the integral

What we do using Riemann sums

- We get a definition of the integral that does not depend on the order in which we do x and y.
- We get a proper definition of volume under the graph.
- We show that continuous functions are integrable.
- We show that continuous functions apart from discontinuities that lie on curves that are the graphs of functions are integrable.
- We prove Fubini's theorem
- We establish formal properties of the integral